Abstract: Orders of reductions of elliptic curves with many and few prime factors

Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication and consider the quantity $\omega(\#E(\mathbb{F}_p))$, where $\omega(n)$ denotes the number of distinct prime factors of n and p is a prime of good reduction for E. Independent work of Cojocaru and Liu shows that the normal order of $\omega(\#E(\mathbb{F}_p))$ is $\log\log p$, and moreover that there is an elliptic curve analogue of the celebrated Erdős - Kac theorem: The quantity

$$\frac{\omega(\#E(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}}$$

has a Gaussian normal distribution. In this talk, we will discuss the frequency with which $\omega(\#E(\mathbb{F}_p))$ is much larger or smaller than expected. For fixed $\gamma > 1$, we have

$$\#\{p \leq x : \omega(\#E(\mathbb{F}_p)) > \gamma \log\log x\} = \frac{x}{(\log x)^{2+\gamma\log\gamma-\gamma+o(1)}}.$$

The same result holds for the quantity $\#\{p \leq x : \omega(\#E(\mathbb{F}_p)) < \gamma \log\log x\}$ when $0 < \gamma < 1$.