Math 8410: Algebraic Number Theory II

This course will cover the theory of local fields, introduce adeles and ideles, and survey the main results of class field theory. These topics are explained in more detail below.

Local Fields: The completion of $\mathbb{Q}$ with respect to the usual absolute value $|x|$ is $\mathbb{R}$. However, there are other absolute values on $\mathbb{Q}$, the so-called p-adic absolute values $|x|_p$, which measure the divisibility of x by a prime p. The completion of $\mathbb{Q}$ with respect to $|x|_p$ is called the field of p-adic numbers. A theorem of Ostrowski says that $\mathbb{R}$ and the fields $\mathbb{Q}_p$ are the only complete, locally compact fields containing $\mathbb{Q}$ as a dense subset. This is reflected in the "product formula": for each $0 \neq x \in \mathbb{Q}$, one has $|x| \cdot \prod_p |x|_p = 1$. In general, local fields K_v are the completions of global fields K (number fields, or a function fields in one variable over a finite field of constants)

Since the p-adic fields are complete and locally compact, they are good domains for doing analytic arguments. A common strategy in studying diophantine questions over $\mathbb{Q}$ is to first study them over $\mathbb{R}$ and the p-adic fields, then collate the results and hopefully obtain an answer over $\mathbb{Q}$. A stellar example is the Hasse-Minkowski theorem: a quadratic form over $\mathbb{Q}$ represents a rational number α if and only if it represents α over $\mathbb{R}$ and all the p-adic fields $\mathbb{Q}_p$.

The first half of the course will be devoted to the theory of local fields. Topics to be covered include the structure of local fields, Hensel's lemma, Ostrowski's theorem, the Hasse-Minkowski Theorem, Haar measures on local fields, and the Galois theory of local fields.

Adeles and Ideles: If one starts with a number field K, it is natural to study the product $\prod_v K_v$ of all its completions. However, it turns out that this product is "too big", and it is better to use a subset of the product called the restricted direct product. The adele ring $\mathbb{A}_K$ is the restricted direct product of the fields K_v, and the idele group $\mathbb{J}_K$ is the restricted direct product of the multiplicative groups $K_v^\times$.

The field K embeds discretely in the adele ring $\mathbb{A}_K$, and the quotient $\mathbb{A}_K/K$ is compact. This fact is equivalent to the combination of Dirichlet's unit theorem and the finiteness of the class number.

Adeles and ideles are important in class field theory, representation theory (the 'Langlands Program'), and in the study of zeta-functions and L-functions. The next third of the course will be devoted to the theory of adeles and ideles.

Class Field Theory: Global Class field theory is a "big hammer" in number theory. It describes the set of all abelian extensions of a number field in terms of objects attached to the number field itself (in the modern formulation, the objects are quotients of the idele group; in the classical formulation, they are quotients of generalized ideal class groups). Applications of class field theory include the higher power reciprocity laws, the Chebotarev density theorem, and Kronecker's theorem which says that each finite abelian extension of $\mathbb{Q}$ is a subfield of some cylotomic extension.

The final sixth of the course will be an introduction to global class field theory (without proofs), explaining what is is, formulating its main theorems, and giving some of its applications.